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Fock–Bargmann representation of the distorted Heisenberg algebra

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Abstract. The dynamical algebra associated with a family of Isospectral Oscillator Hamiltonians, named *distorted Heisenberg algebra* because of its dependence on a distortion parameter $w \geqslant 0$, has recently been studied. The connection of this algebra with the Hilbert space of entire analytic functions of growth $(\frac{1}{2}, 2)$ is analysed.

1. Introduction

In 1980, Abraham and Moses found a general class of one-dimensional potentials isospectral to the oscillator one by means of the Gelfand–Levitan formalism [1]. An elegant way of constructing the same class, used by Mielnik [2], consists of the application of a variant of the standard factorization method. The connection between the Darboux transformation and this generalized factorization has recently been discussed [3–5]. The Mielnik construction is suitable for the easy identification of a pair of annihilation and creation operators for the isospectral oscillator Hamiltonians $\{A, A^\dagger\}$, which are adjoint to each other, although their commutator is not the identity [2]. Departing of the previous operators, a different pair, $\{A, B^\dagger\}$, was constructed such that the commutator is the identity but $(A)^\dagger \neq B^\dagger$ [6]. Recently, a third choice was made [7], where the annihilation C_w and creation $C_w^\dagger$ operators are adjoint to each other and commute with the identity on a subspace of the state space, imitating then the behaviour of the usual annihilation and creation operators for the harmonic oscillator, i.e. the Heisenberg–Weyl algebra. It turns out that C_w and $C_w^\dagger$ depend on a parameter $w \geqslant 0$. The apparence of this parameter is important because it leads to the Heisenberg–Weyl algebra for some of its particular values. Moreover, the coherent states constructed from these operators reach the standard form of the harmonic oscillator coherent states for such w-values. In the general case (w arbitrary), these operators have an algebraic structure very similar to the harmonic oscillator one.

The Fock–Bargmann representation of the Heisenberg–Weyl algebra is widely used in physics and mathematics [8]. The first application was in quantum field theory, where the operators $\bar{z}$ and $\partial/\partial\bar{z}$ represent the creation and annihilation of bosons. Recently, this algebra has also been considered in the study of tensor bosons, arising in composite object theories, which according to their symmetry properties have been classified as symmetric or antisymmetric [9].

The resemblance between the operator pair $\{C_w, C_w^\dagger\}$ and the Heisenberg–Weyl corresponding one $\{a, a^\dagger\}$, suggests the following question, which we will try to answer in

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this paper: what are the properties of the operators C_w and $C_w^\dagger$ when they act on a space of entire analytic functions? In particular, we will consider just one of the two kinds of coherent states obtained by Fernández *et al* [7], and we will show that the corresponding Fock–Bargmann space $\mathcal{F}_w$ contains entire analytic functions of growth $(\frac{1}{2}, 2)$. The realization of C_w and $C_w^\dagger$ on $\mathcal{F}_w$ is a multiplication by $\bar{z}$ and a derivation with respect to $\bar{z}$ plus a new term dependent on the distortion parameter respectively. We also discuss the specific example with $w = 1$, and we show that the usual harmonic oscillator case is recovered on $\mathcal{F}_1$.

2. Distorted Heisenberg algebra revisited

Let us consider an infinite discrete set of orthonormal state vectors in the Hilbert space $\mathcal{H}$,$\{|\theta_n\rangle, n = 0, 1, 2, \ldots\}$. This is a basis set in $\mathcal{H}$ related to the standard harmonic oscillator basis $\{|\psi_n\rangle, n = 0, 1, 2, \ldots\}$ [2] by

$$b|\theta_n\rangle = \sqrt{n}\,|\psi_{n-1}\rangle \qquad b^\dagger|\psi_n\rangle = \sqrt{n+1}\,|\theta_{n+1}\rangle.$$

The $|\theta_n\rangle$'s, $n = 0, 1, 2, \ldots$, satisfy the following eigenvalues equation:

$$H_\lambda|\theta_n\rangle = E_n^{(\lambda)}|\theta_n\rangle$$

where $E_n^{(\lambda)} = E_n = n + \frac{1}{2}$, and $H_\lambda = b^\dagger b + \frac{1}{2}$ is the Hamiltonian isospectral to the harmonic oscillator Hamiltonian $H = a^\dagger a + \frac{1}{2} = aa^\dagger - \frac{1}{2}$ obtained through the generalized factorization method of Mielnik [2]. The explicit expression of H_λ in the coordinate representation is

$$H_\lambda = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{x^2}{2} - \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{e}^{-x^2}}{\lambda + \int_0^x \mathrm{e}^{-y^2}\,\mathrm{d}y}\right].$$

Note that the factorizing operators $b^\dagger$ and b are not the raising and lowering operators for the basis $\{|\theta_n\rangle\}$. According to Fernández *et al* [7], the raising and lowering operators similar to those of the harmonic oscillator, and leading to the distorted Heisenberg–Weyl algebra, are built by means of the condition

$$[C, C^\dagger] = I \qquad \text{on} \qquad \mathcal{H}_s \subset \mathcal{H}$$

where $\mathcal{H}_s$ is a subspace of the Hilbert space $\mathcal{H}$.

It turns out that these operators depend on a parameter $w \geqslant 0$ and take the form [7]

$$C_w = b^\dagger \frac{1}{N+1}\sqrt{\frac{N+w}{N+2}}ab \qquad C_w^\dagger = b^\dagger a^\dagger \frac{1}{N+1}\sqrt{\frac{N+w}{N+2}}b.$$

Let us remark that since b, $b^\dagger$, a, $a^\dagger$ are first-order differential operators, then C_w and $C_w^\dagger$ are differential operators of order greater than three (indeed they are infinite-order differential operators). Their action on the basis $\{|\theta_n\rangle\}$ is given by

$$\begin{aligned} C_w|\theta_n\rangle &= (1 - \delta_{n,0} - \delta_{n,1})\,\sqrt{n-2+w}\,\,|\theta_{n-1}\rangle \\ C_w^\dagger|\theta_n\rangle &= (1 - \delta_{n,0})\,\sqrt{n-1+w}\,\,|\theta_{n+1}\rangle \\ I_w|\theta_n\rangle &= [1 - \delta_{n,0} + \delta_{n,1}(w-1)]|\theta_n\rangle \end{aligned} \tag{2.1}$$

where $I_w \equiv [C_w, C_w^\dagger]$. From this action, it is clear that for $w > 0$ the set of operators $\{C_w, C_w^\dagger, I_w\}$ enables one to decompose the Hilbert space $\mathcal{H}$ as a direct sum of two invariant subspaces, $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_r$, where $\mathcal{H}_0$ is spanned by $|\theta_0\rangle$ and $\mathcal{H}_r$ by $\{|\theta_n\rangle, n \geqslant 1\}$. These two subspaces induce irreducible representations of C_w, $C_w^\dagger$ and I_w. The usual representation of the Heisenberg algebra is recovered on $\mathcal{H}_r$ for $w = 1$. On the other hand, for $w = 0$ $\mathcal{H}$ decomposes (under the action of $\{C_0, C_0^\dagger, I_0\}$) as the direct sum of three invariant subspaces

$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_s$, where $\mathcal{H}_1$ is generated by $|\theta_1\rangle$ and $\mathcal{H}_s$ by $\{|\theta_n\rangle, n \geqslant 2\}$, and the usual Heisenberg algebra representation is once again recovered on $\mathcal{H}_s$. In this paper we will work the case $w > 0$, and when we consider a particular case we will analyse just that with $w = 1$.

We can propose the new operator $N_w = C_w^\dagger C_w$, analogous to the standard number operator $N = a^\dagger a$, the relevant representation of which arises when we consider its action on the vectors $|\psi\rangle \in \mathcal{H}_r$. The standard number operator representation is recovered on $\mathcal{H}_r$ by taking $w = 1$ in N_w and relabelling the eigenstates of H_λ as $|\phi_n\rangle \equiv |\theta_{n+1}\rangle$. However, the dependence of N_w as a function of the Hamiltonian H_λ is not obvious (in the general case $w > 0$). A nonlinear dependence is probably in the context of the recent generalized Fock treatment [10], but it is quite involved to determine it precisely. This is due to the fact that the order of C_w and $C_w^\dagger$ is in general infinity while the one of H_λ is finite (order two).

The normalized ground state of H_λ, $|\theta_0\rangle$, can be introduced by means of the requirements $C_w|\theta_0\rangle = C_w^\dagger|\theta_0\rangle = 0$. The solution to these equations in the coordinate representation is given [6] by

$$\theta_0(x) \propto \frac{\mathrm{e}^{-x^2/2}}{\lambda + \int_0^x \mathrm{e}^{-y^2}\,\mathrm{d}y}$$

where $\lambda \in \mathbb{R}$, $|\lambda| > \sqrt{\pi}/2$. Hence $|\theta_0\rangle$ is orthogonal to all $|\theta_n\rangle \in \mathcal{H}_r$ [2]. Because $C_w|\theta_1\rangle = 0$ (see equation (2.1)), the eigenvalue $z = 0$ of C_w is doubly degenerated. As a consequence, the state $|\theta_0\rangle$ is disconnected of the space $\mathcal{H}_r$ and $|\theta_1\rangle$ is the state playing the role of the extremal state for the distorted Heisenberg–Weyl algebra. Thus, the operator $C_w^\dagger$ can be used to construct the basis of the state space $\mathcal{H}_r$ from its repeated action on $|\theta_1\rangle$:

$$|\theta_n\rangle = \sqrt{\frac{\Gamma(w)}{\Gamma(w+n-1)}}(C_w^\dagger)^{n-1}|\theta_1\rangle \qquad w \neq 0. \tag{2.2}$$

A similar situation has been recently reported by Spiridonov for systems and creation and annihilation operators different to the ones used in this paper (see [4, section VII]). Spiridonov also looks for the coherent states as eigenstates of the corresponding annihilation operator, and they could be found through the solution of some third-order differential equations ([4, equations (7.10) and (7.13)]). Unfortunately, one cannot find those coherent states easily or directly be means of such a method; however, it can be done for the isospectral Hamiltonians with which we are dealing [6, 7]. This is one of the advantages of the coherent states we will present in the next section.

As a final point of this section, the resolution of the identity in terms of the basis $\{|\theta_n\rangle, n \geqslant 0\}$ is the standard one

$$I = \sum_{n=0}^{\infty} |\theta_n\rangle\langle\theta_n|.$$

This lets us expand any state vector $|h\rangle \in \mathcal{H}$ as

$$|h\rangle = \sum_{n=0}^{\infty} a_n|\theta_n\rangle \qquad a_n \equiv \langle\theta_n|h\rangle.$$

3. Distorted Heisenberg algebra coherent states

Recently, a family of coherent states has been constructed as eigenstates of the annihilation operator, $C_w|z, w\rangle = z|z, w\rangle$; here they are called *w-coherent states*. Their explicit form in terms of $|\theta_n\rangle$ is given [7] by

$$|z, w\rangle = \sqrt{\frac{\Gamma(w)}{{}_1F_1(1, w; r^2)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\Gamma(w+n)}} |\theta_{n+1}\rangle \qquad z = r\mathrm{e}^{\mathrm{i}\varphi} \tag{3.1}$$

where ${}_1F_1(a, b; x)$ is a hypergeometric function. The ket $|\theta_0\rangle$, by construction, is also a coherent state. The set $\{|\theta_0\rangle, |z, w\rangle\}$ is complete in $\mathcal{H}$, with an identity resolution

$$I = |\theta_0\rangle\langle\theta_0| + \int |z, w\rangle\langle z, w| \, \mathrm{d}\mu(z, w) \tag{3.2}$$

where $\mathrm{d}\mu(z, w)$ is the measure written as

$$\mathrm{d}\mu(r, w) = \frac{{}_1F_1(1, w; r^2)}{\pi\Gamma(w)} \, \mathrm{e}^{-r^2} \, r^{2(w-1)} \, r\mathrm{d}r \, \mathrm{d}\varphi. \tag{3.3}$$

As usual, this new basis set is not orthogonal because the inner product between two different coherent states

$$\langle z, w|z', w\rangle = \frac{{}_1F_1(1, w; \bar{z}z')}{\sqrt{{}_1F_1(1, w; r^2) \, {}_1F_1(1, w; r'^2)}} \tag{3.4}$$

is in general, different from zero. Any vector state $|h\rangle \in \mathcal{H}$ is now expanded as

$$|h\rangle = h_0|\theta_0\rangle + \int \tilde{h}(z, \bar{z}, w)|z, w\rangle \, \mathrm{d}\mu(z, w) \tag{3.5}$$

where $h_0 \equiv \langle\theta_0|h\rangle$

$$\tilde{h}(z, \bar{z}, w) \equiv \langle z, w|h\rangle = \frac{h_w(\bar{z})}{\sqrt{{}_1F_1(1, w; r^2)}} \tag{3.6}$$

and

$$h_w(\bar{z}) \equiv \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(w)}{\Gamma(w+n)}} \bar{z}^n \langle\theta_{n+1}|h\rangle. \tag{3.7}$$

In this representation, the inner product between $|f\rangle \in \mathcal{H}$ and $|g\rangle \in \mathcal{H}$ is

$$\begin{aligned} \langle f|g\rangle &= \bar{f}_0 g_0 + \int \frac{\bar{f}_w(\bar{z}) g_w(\bar{z})}{{}_1F_1(1, w, r^2)} \, \mathrm{d}\mu(\bar{z}, w) \\ &= \bar{f}_0 g_0 + \int \bar{f}_w(\bar{z}) g_w(\bar{z}) \, \mathrm{d}\sigma_w(\bar{z}) \end{aligned} \tag{3.8}$$

where

$$\mathrm{d}\sigma_w(z) = \frac{\mathrm{e}^{-r^2} \, r^{2(w-1)}}{\pi\Gamma(w)} \, r\mathrm{d}r \, \mathrm{d}\varphi. \tag{3.9}$$

In the special case $f = g$ we obtain

$$|| \, |f\rangle \, ||^2 = |f_0|^2 + \int |f_w(\bar{z})|^2 \, \mathrm{d}\sigma_w(z) \geqslant 0. \tag{3.10}$$

4. The Hilbert space of entire functions

4.1. Basic properties of $\mathcal{F}_w$

The goal of this section is to characterize any vector state by an entire analytic function. Such a realization of the Hilbert space $\mathcal{H}$ is called the *Fock–Bargmann representation* [11, 12], and it can be obtained through any coherent state system. Here, we will consider the coherent states $|z, w\rangle$ and the Hilbert space $\mathcal{H}_r \subset \mathcal{H}$.

Let us introduce the Hilbert space $\mathcal{F}_w$, which elements are all functions of the form (3.7). For each w, the inner product is defined by

$$(f, g)_w \equiv \int \bar{f}_w(\bar{z}) g_w(\bar{z})\, \mathrm{d}\sigma_w(\bar{z}) \tag{4.1}$$

where the integral is extended over all $\mathbb{C}$. In particular, for $f = g$ we require (see equation (3.10))

$$(f, f)_w = \int |f_w(\bar{z})|^2\, \mathrm{d}\sigma_w(\bar{z}) < \infty. \tag{4.2}$$

It can easily be seen that $f_w(z)$ is an entire analytic function (see equation (3.7)). We want to study in more detail the properties of $f_w(z)$. With this aim, let us analyse equation (3.6). Because $|z, w\rangle$ is normalized, the Schwarz inequality gives $|\langle z, w|f\rangle| \leqslant ||\,|f\rangle\,||$, $|f\rangle \in \mathcal{H}_r$, and one obtains

$$|f_w(z)| \leqslant \sqrt{{}_1F_1(1, w, r^2)} \quad ||\,|f\rangle\,||. \tag{4.3}$$

Hence the behaviour of $|f_w(z)|$ at infinity is the same as that of $\sqrt{{}_1F_1(1, w, r^2)}$. As has been shown, the generalized hypergeometric function

$$\begin{aligned} {}_pF_q(a_1, \ldots, a_p, b_1, \ldots, b_q, z^s) &\equiv {}_pF_q(a_i, b_j, z^s) \\ &= \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)} \sum_{n=0}^{\infty} \frac{\Gamma(a_1+n)\cdots\Gamma(a_p+n)}{\Gamma(b_1+n)\cdots\Gamma(b_q+n)} \frac{z^n}{n!} \end{aligned} \tag{4.4}$$

is an entire analytic function (exponential type) of order ρ_s and type σ_s [6], i.e. growth (σ_s, ρ_s). The relation between ρ_s and σ_s is given by $\sigma_s = s/\rho_s = 1 + q - p$. In the particular case of ${}_1F_1(1, w, r^2)$ we find $\rho_s = 2$ and $\sigma_s = 1$. Then, because

$$|\sqrt{{}_pF_q(a_i, b_j; z^s)}| = \sqrt{|{}_pF_q(a_i, b_j, z^s)|} \leqslant \exp\left(\tfrac{1}{2}\sigma_s r^{\rho_s}\right) \tag{4.5}$$

one obtains that $\sqrt{{}_1F_1(1, w, r^2)}$ has $\rho = 2$ and $\sigma = \frac{1}{2}$, i.e. growth $(\frac{1}{2}, 2)$. This behaviour at infinity is equal to the corresponding one for the harmonic oscillator case, which shows that the w coherent states are 'good' for generating the Fock–Bargmann representation of $\mathcal{H}_r$. Thus, we have realized the Hilbert space $\mathcal{H}_r$ as a space $\mathcal{F}_w$ of entire analytic functions $f_w(z)$ of the form (3.7) and satisfying condition (4.2).

The orthonormal basis $\{|\theta_n\rangle, n \geqslant 1\}$ in $\mathcal{F}_w$ has a simple form

$$\theta_n(\bar{z}) \equiv \sqrt{{}_1F_1(1, w, r^2)}\ \ \langle z, w|\theta_n\rangle = \bar{z}^{(n-1)} \sqrt{\frac{\Gamma(w)}{\Gamma(w+n-1)}}. \tag{4.6}$$

This is the simplest orthonormal set of vectors in $\mathcal{F}_w$. The functions $h_w(\bar{z})$ are then written as

$$h_w(\bar{z}) = \sum_{n=0}^{\infty} A_{n+1}\theta_{n+1}(\bar{z}) \qquad A_{n+1} \equiv \langle \theta_{n+1}|h\rangle \tag{4.7}$$

and the corresponding representation of the coherent state $|\alpha, w\rangle$, $\alpha \in \mathbb{C}$ is

$$\alpha_w(\bar{z}) = \sqrt{{}_1F_1(1, w, r^2)}\ \langle z, w|\alpha, w\rangle$$
$$= \frac{{}_1F_1(1, w, \bar{z}\alpha)}{\sqrt{{}_1F_1(1, w, |\alpha|^2)}}. \tag{4.8}$$

Note that equations (4.6) and (4.8) reproduce the usual harmonic oscillator case when we take $w = 1$.

4.2. Principal vectors $\boldsymbol{e}_a$, and the reproducing kernel

Equation (4.3) is quite typical for a Hilbert space of analytic functions, and the term $\sqrt{{}_1F_1(1, w, r^2)}$ arises due to the set of coherent states taken into account. In this section we will derive some results independent of the function ${}_1F_1(1, w, r^2)$ and valid for all possible spaces $\mathcal{F}$.

Let us consider the elements $\boldsymbol{e}_a$ of $\mathcal{F}_w$ (for every $a \in \mathbb{C}$) given by

$$\boldsymbol{e}_a(z) \equiv \sum_{n=0}^{\infty} \theta_{n+1}(z)\bar{\theta}_{n+1}(a) = {}_1F_1(1, w, z\bar{a}). \tag{4.9}$$

Using these vectors, called the *principal vectors* of $\mathcal{F}_w$, we can introduce a bounded linear functional

$$f_w(a) = (\boldsymbol{e}_a, f)_w. \tag{4.10}$$

The integral form reads

$$f_w(a) = \int {}_1F_1(1, w, a\bar{z}) f_w(z)\, \mathrm{d}\sigma_w(z). \tag{4.11}$$

Then, ${}_1F_1(1, w, a\bar{z})$ is the reproducing kernel for $\mathcal{F}_w$. Conversely, equation (4.10) implies (4.3) with $||\boldsymbol{e}_a|| = {}_1F_1(1, w, |a|^2)$.

The connection between the reproducing kernel and the coherent states as represented in the $\mathcal{F}_w$ space is given by (4.8). This could be seen also by expressing (4.11) in terms of the w coherent states $|z, w\rangle \in \mathcal{H}_r$ because

$$f_w(\bar{a}) = \sqrt{{}_1F_1(1, w, |a|^2)}\langle a, w|f\rangle$$
$$= \int \frac{{}_1F_1(1, w, \bar{a}z)}{\sqrt{{}_1F_1(1, w, r^2)}} \langle z, w|f\rangle\, \mathrm{d}\mu(z, w)$$
$$= \int {}_1F_1(1, w, z\bar{a}) f_w(\bar{z})\, \mathrm{d}\sigma_w(z).$$

From this, we get the proportionality between the reproducing kernel and the inner product of two w coherent states $|a, w\rangle$ and $|z, w\rangle$ both in $\mathcal{H}_r$.

Finally, the set of vectors $\boldsymbol{e}_a$ is complete, i.e. their finite linear combinations are dense in $\mathcal{F}_w$, because the only vector orthogonal to all of them is $f = 0$, as follows immediately from (4.10).

4.3. Realization of the distorted Heisenberg algebra in $\mathcal{F}_w$

Let us represent now the generators of the distorted algebra in $\mathcal{F}_w$, which is a subspace of the space of entire functions of growth $(\frac{1}{2}, 2)$. At this point, it is convenient to introduce the unnormalized w coherent states $|z, w\rangle_e = \sqrt{{}_1F_1(1, w, r^2)}|z, w\rangle$. In terms of these the function $h_w(\bar{z})$ can be written as

$$h_w(\bar{z}) = {}_e\langle z, w|h\rangle.$$

The action of $C_w^\dagger$ on $h_w(\bar{z})$ arises from the inner product of $|h\rangle$ with the adjoint of $C_w|z, w\rangle_e = z|z, w\rangle_e$:

$$C_w^\dagger h_w(\bar{z}) \equiv {}_e\langle z, w|C_w^\dagger|h\rangle = \bar{z}h_w(\bar{z}). \tag{4.12}$$

The action of C_w on $h_w(\bar{z})$ is also obtained from the inner product of $|h\rangle$ with the adjoint of

$$C_w^\dagger|z, w\rangle_e = \sum_{m=1}^{\infty} \sqrt{\frac{\Gamma(w)}{\Gamma(w+m)}} z^{m-1} m|\theta_{m+1}\rangle + \frac{(w-1)}{z} \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(w)}{\Gamma(w+n)}} z^n |\theta_{n+1}\rangle$$
$$- \frac{(w-1)}{z}|\theta_1\rangle \,. \tag{4.13}$$

We then get

$$C_w h_w(\bar{z}) = \frac{\partial}{\partial \bar{z}} h_w(\bar{z}) + \frac{(w-1)}{\bar{z}} \left(h_w(\bar{z}) - h_1\right) \tag{4.14}$$

where $h_1 \equiv \langle\theta_1|h\rangle$, $\bar{z} \neq 0$. Finally, the action of I_w on $h(\bar{z})$ is given by

$$I_w h_w(\bar{z}) = (w-1)h_1 + h_w(\bar{z}). \tag{4.15}$$

At this point we see the great resemblance, almost equality, of equations (4.12), (4.14), (4.15), with the corresponding ones for the harmonic oscillator case. The difference rests on the dependence on $w > 0$. When $w = 1$, however, equations (4.12), (4.14), (4.15) are the same as those for the usual harmonic oscillator in the Segal–Bargmann space. There is then an isomorphism between $\mathcal{F}_w$ and $\mathcal{H}_r$ for all $w > 0$. Note that it is also possible to find an isomorphism between $\mathcal{F}_w$ and a subspace of $\mathcal{H}$ for $w = 0$, but in this case it will be $\mathcal{H}_s$.

We would finally like to show that the coherent states $\alpha_w(z) \in \mathcal{F}_w$, expressed in (4.8) in terms of $|z, w\rangle \in \mathcal{H}_r$, can be obtained through a standard procedure [6, 7] as eigenfunctions of C_w in $\mathcal{F}_w$:

$$C_w\alpha_w(\bar{z}) = \alpha\ \alpha_w(\bar{z}) \qquad \alpha \in \mathbb{C}. \tag{4.16}$$

Let us take

$$\alpha_w(\bar{z}) = \sum_{n=1}^{\infty} a_n\theta_n(\bar{z}). \tag{4.17}$$

Using equation (4.14), with $h_w(\bar{z}) = \alpha_w(\bar{z})$ and (4.16) one obtains the coefficients a_n. In the case $w \neq 0$ we get

$$a_{n+1} = \alpha^n \sqrt{\frac{\Gamma(w)}{\Gamma(w+n)}} a_1 \qquad n = 1, 2, \ldots . \tag{4.18}$$

Then, by imposing the normalization, with $a_1 \geqslant 0$, we obtain

$$\alpha_w(\bar{z}) = \frac{{}_1F_1(1, w, \alpha\bar{z})}{\sqrt{{}_1F_1(1, w, |\alpha|^2)}}$$

which, obviously coincides with (4.8). Note that the case $w = 1$ is straightforwardly obtained from the previous formula:

$$\alpha_1(\bar{z}) = \mathrm{e}^{(-\frac{1}{2}|\alpha|^2 + \alpha\bar{z})}.$$

This is the usual representation for the harmonic oscillator coherent state in the Segal–Bargmann space.

All these facts show that C_w and $C_w^\dagger$, adjoint to each other in $\mathcal{H}_r$, possess the same properties on $\mathcal{F}_w$. Hence, the eigenstates of C_w in $\mathcal{H}_r$ correspond to eigenfunctions of C_w in $\mathcal{F}_w$ and vice versa.

5. Concluding remarks

The w coherent states $|z, w\rangle$ and the operators C_w, $C_w^\dagger$ and I_w, admit a simplest representation in the space of analytic functions $\mathcal{F}_w$ compared with the representation in $\mathcal{H}_r$. The usual harmonic oscillator representation is achieved when $w = 0$ or $w = 1$. The fact that C_w and $C_w^\dagger$ are adjoint to each other on $\mathcal{H}_r$ is preserved on $\mathcal{F}_w$ because the isomorphism between $\mathcal{F}_w$ and $\mathcal{H}_r$. This representation of the distorted Heisenberg algebra on $\mathcal{F}_w$ as far as we know, has not been explored previously. It could be important in the analysis of the appropriate displacement operator D_w to perform the Perelomov construction of the coherent states for the isospectral oscillator Hamiltonians H_λ.

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